Vacuum Energy in Ultralocal metrics for TT tensors with Gaussian Wave Functionals

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Abstract

We calculate, in a class of Gauge invariant functionals, by variational methods, the difference of vacuum energy between two different backgrounds: Schwarzschild and Flat Space. We perform this evaluation in an Hamiltonian formulation of Quantum Gravity by standard "3 + 1" decomposition. After the decomposition the scalar curvature is expanded to second order with respect to the Schwarzschild metric. We evaluate this energy difference in momentum space, in the lowest possible state (regardless of any negative mode). We find a singular behaviour in the UV- limit, due to the presence of the horizon when $r = 2m$. When $r > 2m$ this singular behaviour disappears, which is in agreement with various other models presented in the literature.

I. INTRODUCTION

The problem of computing quantum corrections to a classical energy in a complicated theory such as Einstein gravity, can be approached by performing an analysis of the thermodynamical quantities that characterize the system under consideration. This analysis can be done by means of the computation of the free energy of the system at a given volume and temperature. Defining the Euclidean action as

$$\hat{I}\left[g\right] = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}R\left(g\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K^i_i, \tag{1}$$

where R is the Ricci scalar of the metric $g_{\mu\nu}$ and K^i_i is the trace of the second fundamental form, one can compute quantum corrections to the Euclidean action, of a fixed background geometry. A background of this type is termed asymptotically flat, which means that the metric approaches the flat metric $R^3 \times S^1$ outside some compact set. The boundary of infinity is topologically $S^2 \times S^1$. Let us consider a different point of view, and precisely the hamiltonian one. In this framework one is able to deal with three dimensional fields configurations separated out by the time variable. Obviously, the topology of the boundary at infinity is quite different in this case, because it is exactly $S^2 \times R$, while the metric approaches the flat metric $R^3 \times R^1$ asymptotically. The advantage of the Hamiltonian framework is that one can manage from the beginning with energy fields configurations which, give directly the quantum corrections to the classical term. The first step, in this hamiltonian approach, is the separation of the space-time line element in 3 space plus 1 time. For this purpose we introduce the $\mathcal{ADM}$ variables [4], (shift and lapse functions), such that the space-time line element becomes

$$\begin{aligned} ds^2 = g_{\mu\nu}\left(x\right)dx^\mu dx^\nu &= -N^2\left(dx^0\right)^2 + g_{ij}\left(N^i dx^0 + dx^i\right)\left(N^j dx^0 + dx^j\right) \\ &= \left(-N^2 + N_i N^i\right)\left(dx^0\right)^2 + 2N_j dx^0 dx^j + g_{ij}dx^i dx^j, \end{aligned} \tag{2}$$

where the matrix representation of $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N_j \\ N_i & g_{ij} \end{pmatrix},$$

with the inverse given by (3)

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & g_{ij} - \frac{N_i N^i}{N^2} \end{pmatrix}.$$

Roman indices will be raised and lowered by the induced metric on the three surface x^0. In terms of the $\mathcal{ADM}$ variables, the initial action can be written as a sum of a "*kinetic*" and a "*potential*" term

$$I = \frac{1}{16\pi G}\int dx^0 N \int dx^3\ ^{(3)}\sqrt{g}\left\{\left(K_{ij}K^{ij} - K^2\right) +\ ^{(3)}R\right\}, \tag{4}$$

where $K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - g_{ij,0}\right)$ is called the second fundamental form and "$|$ " means covariant differentiation with respect to the $3D$ gravitational background, K is the trace of the second fundamental form, ^3R is the scalar curvature in $3D$, $^{(3)}\sqrt{g}$ is the invariant of the metric in $3D$. In this form the "*time derivative*" is isolated and it is possible the computation of the conjugate momentum to g_{ij},that is

$$\pi^{ij} = \frac{\delta I}{\delta\ \dot{g_{ij}}} = \left(-K^{ij} +\ ^{(3)}g^{ij}K\right)\frac{\sqrt{g}}{16\pi G}. \tag{5}$$

By a Legendre transformation we calculate the Hamiltonian

$$\begin{aligned} H = \textstyle\int d^3x \left\{\left[\pi^{ij}\ \dot{g_{ij}} - \frac{1}{16\pi G}\left[\left(K_{ij}K^{ij} - K^2\right) +\ ^{(3)}R\right] N\ ^{(3)}\sqrt{g}\right]\right\} = \\ \int d^3x \left\{N\left[\frac{16\pi G}{^{(3)}\sqrt{g}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{2}\right) -\ ^{(3)}\sqrt{g}\frac{^{(3)}R}{16\pi G}\right] + N_i\left(2\pi^{ij}_{|j}\right)\right\}. \end{aligned} \tag{6}$$

Looking at the first term, we can see that it has a quadratic structure in the momenta. This suggests us that as a first approximation we could compute quantum corrections to the energy expanding ^3R in terms of the quantum fluctuations of the fields with respect to a given background, i.e. Schwarzschild. It is clear that, in this picture, because only the spatial part of this fixed geometry is relevant, one can speak about the computation of quantum correction of the energy in a wormhole background. The rest of the paper is structured as follows, in section II we analyze the orthogonal decomposition of the hamiltonian both in tangent and co-tangent space, in section III we define the gaussian wave functional for gravity in analogy with non-abelian gauge theories, in section IV we give some of the basic rules to perform the functional integration and we define the Hamiltonian approximated up to second order, in section V, we analyze the spin-2 operator or the operator acting on transverse traceless tensors, only for positive values of E^2. We summarize and conclude in section VI.

II. ULTRALOCAL METRICS AND THE HAMILTONIAN

After the introduction of $\mathcal{ADM}$

variables, we recall that the Hamiltonian is:

$$H = \int d^3x (N\mathcal{H}+N_i\mathcal{H}^i) \tag{7}$$

where

$$\mathcal{H}=G_{ijkl}\pi^{ij}\pi^{kl}\left(\frac{l_p^2}{\sqrt{g}}\right)-\left(\frac{\sqrt{g}}{l_p^2}\right)\ {}^{(3)}R\ \ \text{(Super Hamiltonian)} \tag{8}$$

and

$$\mathcal{H}^i = -2\pi^{ij}_{|j}\ \ \text{(Super Momentum)}. \tag{9}$$

In (9) the derivative is covariant with respect to the $3D$ background field, l_p^2 is the usual Planck mass, and G_{ijkl} is the Wheeler-DeWitt (WDW) metric. If we look at N and N_i as fundamental objects describing the correct variables, by variational principles we obtain the usual constraint equations, that is

$$\begin{aligned} \mathcal{H}\ &= 0,\ \mathcal{H}^i\ = 0 \text{ Classical} \\ \mathcal{H}\Psi\ &= 0,\ \mathcal{H}^i\ \Psi = 0 \text{ Quantum} \end{aligned}$$

The usual interpretation of these equations is that they represent constraints on the initial value problem or in other words they represent gauge invariance with respect to time and gauge transformations. In this paper we will follow a different approach: we will consider constant lapse function. This choice is the most appropriate for wormholes configuration of the background geometry such as this case [1]. The choice $N = const$ is completely equivalent to rescale time intervals, therefore

$$N = 1 \tag{10}$$

Then the Hamiltonian in the time-like gauge is

$$H = \int d^3x \mathcal{H} \; = \int d^3x \left[G_{ijkl} \pi^{ij} \pi^{kl} \left(\frac{l_p^2}{\sqrt{g}} \right) - \left(\frac{\sqrt{g}}{l_p^2} \right) \; {}^{(3)}R \right] \tag{11}$$

Instead of performing calculations in the usual WDW metric we will use a one-parameter family of super-metrics to disentangle gauge modes from physical deformations. For this reason we require an orthogonal decomposition for both π_{ij} and h_{ij}, that is we need a metric on the space of deformations, i.e. a quadratic form on the tangent space at h. The condition of ultralocality, where G_{ijkl} locally depends on g_{ij} but not on its derivatives, could be taken as a good condition for the functional measure, explicitly:

$$\langle h, k \rangle := \int_{\mathcal{M}} \sqrt{g} G_{\alpha}^{ijkl} h_{ij}(x) \, k_{kl}(x) \, d^3x, \tag{12}$$

where

$$G_{\alpha}^{ijkl} = (g^{ik}g^{jl} + g^{il}g^{jk} - 2\alpha g^{ij}g^{kl}). \tag{13}$$

The WDW metric, introduced in (8), is just (13) with $\alpha = 1$. The "inverse" metric is defined on co-tangent space and it assumes the form

$$\langle p, q \rangle := \int_{\mathcal{M}} \sqrt{g} G_{ijkl}^{\beta} p^{ij}(x) \, q^{kl}(x) \, d^3x, \tag{14}$$

where

$$G_{ijkl}^{\beta} = (g_{ik}g_{jl} + g_{il}g_{jk} - 2\beta g_{ij}g_{kl}). \tag{15}$$

with $\alpha + \beta = 3\alpha\beta$, so that

$$G_{\beta}^{ijnm} G_{nmkl}^{\beta} = \frac{1}{2} \left(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j \right). \tag{16}$$

These are non-degenerate bilinear forms for $\alpha \neq \frac{1}{3}$, for $\alpha = \frac{1}{3}$ the metric is not invertible and becomes a projector onto the tracefree subspace, while is positive definite for $\alpha < \frac{1}{3}$ and of mixed signature for $\alpha > \frac{1}{3}$ with infinitely many plus as well as minus signs.

We have now the desired decomposition on the tangent space of 3-metric deformations h_{ij}:

$$h_{ij} = \frac{1}{3} h g_{ij} + (L\xi)_{ij} + h_{ij}^{\perp}, \tag{17}$$

or, in matrix form,

$$h = \frac{1}{3} hg + (RangeL) + \left(KerL^{\dagger}\right), \tag{18}$$

where the operator L maps ξ_i into symmetric tracefree tensors, according to [2] [3],

$$(L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi). \tag{19}$$

Consequently, the inversion of the metric (14) (that is (15)) ,guarantees us the same decomposition also in phase space (co-tangent space).

III. THE GAUSSIAN WAVE FUNCTIONAL

There are some reasons to introduce a gaussian wave functional for the description of the vacuum state in gravity. People, usually, are more familiar with Non-abelian Gauge theories, where the trial gaussian wave functional is defined by [5]

$$\Psi \left[A_i^a \left(\overrightarrow{x}\right)\right] = \mathcal{N} \exp \left\{-\frac{1}{4} \int d^3x d^3y \delta A_i^a \left(\overrightarrow{x}\right) G_{ij}^{-1ab} \left(\overrightarrow{x}, \overrightarrow{y}\right) \delta A_j^b \left(\overrightarrow{y}\right)\right\} \tag{20}$$

where $\mathcal{N}$ is a normalization factor and where

$$\delta A_i^a \left(\overrightarrow{x}\right) = A_i^a \left(\overrightarrow{x}\right) - \overline{A_i^a} \left(\overrightarrow{x}\right). \tag{21}$$

In equation (21), $\overline{A_i^a} \left(\overrightarrow{x}\right)$ is a background field which can be treated as a variational parameter together to the function $G_{ij}^{ab} \left(\overrightarrow{x}, \overrightarrow{y}\right)$in (20).

From the definition in (20)one finds the expectation values

$$\left\langle \Psi \right| A_i^a \left(\overrightarrow{x} \right) \left| \Psi \right\rangle = \overline{A_i^a} \left(\overrightarrow{x} \right),$$

$$\left\langle \Psi \right| A_i^a \left(\overrightarrow{x} \right) A_j^b \left(\overrightarrow{y} \right) \left| \Psi \right\rangle = \overline{A_i^a} \left(\overrightarrow{x} \right) \overline{A_j^b} \left(\overrightarrow{y} \right) + G_{ij}^{ab} \left(\overrightarrow{x}, \overrightarrow{y} \right),$$

$$\left\langle \Psi \right| E_i^a \left(\overrightarrow{x} \right) \left| \Psi \right\rangle = 0, \tag{22}$$

$$\left\langle \Psi \right| E_i^a \left(\overrightarrow{x} \right) E_j^b \left(\overrightarrow{y} \right) \left| \Psi \right\rangle = \tfrac{1}{4} G_{ij}^{-1ab} \left(\overrightarrow{x}, \overrightarrow{y} \right),$$

$$\left\langle \Psi \right| B_i^a \left(\overrightarrow{x} \right) \left| \Psi \right\rangle = \overline{B_i^a} \left(\overrightarrow{x} \right) + \tfrac{1}{2} \epsilon_{ijk} f^{abc} G_{ij}^{ab} \left(\overrightarrow{x}, \overrightarrow{y} \right),$$

where

$$E_i^a \left(\overrightarrow{x} \right) = -i \frac{\delta}{\delta A_i^a \left(\overrightarrow{x} \right)}, \tag{23}$$

and

$$B_i^a \left(\overrightarrow{x} \right) = \epsilon_{ijk} \left\{ \nabla_j A_k^a \left(\overrightarrow{x} \right) + \frac{1}{2} f^{abc} A_i^a \left(\overrightarrow{x} \right) A_j^b \left(\overrightarrow{y} \right) \right\}, \tag{24}$$

ϵ_{ijk} is the usual anti-symmetric tensor and f^{abc} the structure constants of the gauge group, for ex. $SU(N)$. From the experience on non-abelian gauge theories, we are, now, motivated in defining a "Vacuum Trial State" for gravity, and for these purposes we recall the orthogonal decomposition (16) to look at the essential structure of the inner product between three-geometries

$$\langle h, h \rangle := \int_{\mathcal{M}} \sqrt{g} G_{\alpha}^{ijkl} h_{ij}(x) h_{kl}(x) d^3x =$$

$$\int_{\mathcal{M}} \sqrt{g} \left[\left(\frac{1}{3} - \alpha \right) h^2 + (L\xi)^{ij} (L\xi)_{ij} + h^{ij\perp} h_{ij}^{\perp} \right] \tag{25}$$

Previous formula leads us towards the definition of the "*possible*" trial wave functional for the gravitational ground state

$$\Psi_\alpha \left[h_{ij} \left(\overrightarrow{x} \right) \right] = \mathcal{N} \exp \left\{ -\frac{1}{4 l_p^2} \left[\left\langle h K^{-1} h \right\rangle_{x,y}^{\perp} + \left\langle (L\xi) K^{-1} (L\xi) \right\rangle_{x,y}^{||} + \left\langle h K^{-1} h \right\rangle_{x,y}^{Trace} \right] \right\}, \tag{26}$$

or in other terms

$$\Psi_\alpha\left[h_{ij}\left(\overrightarrow{x}\right)\right]=\mathcal{N}\Psi_\alpha\left[h_{ij}^{\perp}\left(\overrightarrow{x}\right)\right]\Psi_\alpha\left[(L\xi)_{ij}\right]\Psi_\alpha\left[\frac{1}{3}g_{ij}h\left(\overrightarrow{x}\right)\right]. \tag{27}$$

In (26) and in (27), $h_{ij}^{\perp}$ is the tracefree-transverse part of the $3D$ quantum field, $(L\xi)_{ij}$ is the longitudinal part and finally h is the trace part of the same field. The dependence of the functional by α will not be discussed in this paper. The orthogonal decomposition of the gravitational perturbations in its three parts suggest us to choose $\alpha=0$.

In (26), $\langle\cdot,\cdot\rangle_{x,y}$ denotes space integration and K^{-1} is the inverse propagator. The main reason for a similar "*Ansatz*" comes not only from (25) but even to the observation that the momenta quadratic part of the Hamiltonian decouples in the same way. Even if we had to give up to (26) from the beginning, making a more general "*Ansatz*" about the vacuum wave functional (and for more general we mean eqn. (20)) one would discover that the kinetic part decouples in these three terms. For completeness, we give the analogous expectation values for TT tensors. The other components satisfy the same rules

$$\left\langle\Psi|g_{ij}^{\perp}\left(\overrightarrow{x}\right)|\Psi\right\rangle=\bar{g}_{ij}^{\perp}\left(\overrightarrow{x}\right),$$

$$\left\langle\Psi|g_{ij}^{\perp}\left(\overrightarrow{x}\right)g_{kl}^{\perp}\left(\overrightarrow{y}\right)|\Psi\right\rangle=\bar{g}_{ij}^{\perp}\left(\overrightarrow{x}\right)\bar{g}_{kl}^{\perp}\left(\overrightarrow{y}\right)+K_{ijkl}^{\perp}\left(\overrightarrow{x},\overrightarrow{y}\right),$$

$$\left\langle\Psi|\pi_{ij}^{\perp}\left(\overrightarrow{x}\right)|\Psi\right\rangle=0, \tag{28}$$

$$\left\langle\Psi|\pi_{ij}^{\perp}\left(\overrightarrow{x}\right)\pi_{kl}^{\perp}\left(\overrightarrow{y}\right)|\Psi\right\rangle=\frac{1}{4}K_{ijkl}^{-1}\left(\overrightarrow{x},\overrightarrow{y}\right),$$

where $\pi_{ij}^{\perp}=-i\frac{\delta}{\delta h_{ij}^{\perp}(x)}$ is the representation for the TT momentum.

IV. ENERGY DENSITY CALCULATION IN SCHRÖDINGER REPRESENTATION

To calculate the energy density associated to the trial functional, we need to know the action of some basic operators on $\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right]$. The action of the operator h_{ij}on $|\Psi\rangle=\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right]$ is realized by

$$h_{ij}(x)\,|\Psi\rangle = h_{ij}(x)\,\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right] \tag{29}$$

The action of the operator π_{ij}on $|\Psi\rangle$, in general, is

$$\pi_{ij}(x)\,|\Psi\rangle = -i\frac{\delta}{\delta h_{ij}(x)}\Psi\left[h_{ij}\left(\overrightarrow{x}\right)\right]. \tag{30}$$

The inner product is defined by functional integration:

$$\langle\Psi_1 \mid \Psi_2\rangle = \int\left[\mathcal{D}h_{ij}(x)\right]\Psi_1^*\left\{h_{ij}\right\}\Psi_2\left\{h_{kl}\right\}, \tag{31}$$

and energy eigenstates satisfy the Schrödinger equation:

$$\int d^3x\mathcal{H}\left\{-i\frac{\delta}{\delta h_{ij}(x)}, h_{ij}(x)\right\}\Psi\left\{h_{ij}\left(\overrightarrow{x}\right)\right\} = E\Psi\left\{h_{ij}\left(\overrightarrow{x}\right)\right\}, \tag{32}$$

where $\mathcal{H}\left\{-i\frac{\delta}{\delta h_{ij}(x)}, h_{ij}(x)\right\}$ is the Hamiltonian density. The time development is

$$\Psi\left\{h_{ij}\left(\overrightarrow{x}\right);t\right\} = e^{-iEt}\Psi\left\{h_{ij}\left(\overrightarrow{x}\right)\right\}. \tag{33}$$

Instead of solving (32), which is of course impossible, we can formulate the same problem by mean of a variational principle. We demand that

$$\frac{\langle\Psi \mid H \mid \Psi\rangle}{\langle\Psi \mid \Psi\rangle} = \frac{\int\left[\mathcal{D}g_{ij}^{\perp}(x)\right]\int d^3x\Psi_1^*\left\{g_{ij}^{\perp}\right\}\mathcal{H}\Psi\left\{g_{kl}^{\perp}\right\}}{\int\left[\mathcal{D}g_{ij}^{\perp}(x)\right] \mid \Psi\left\{g_{ij}^{\perp}\right\} \mid^2} \tag{34}$$

be stationary against arbitrary variations of $\Psi\left\{h_{ij}\left(\overrightarrow{x}\right)\right\}$. The form of $\langle\Psi \mid H \mid \Psi\rangle$ can be computed as follows. We define normalized mean values by a straightforward modification of (28), i.e.

$$\bar{g}_{ij}^{\perp}(x) = \frac{\int\left[\mathcal{D}g_{ij}^{\perp}(x)\right]\int d^3x g_{ij}^{\perp}(x) \mid \Psi\left\{g_{ij}^{\perp}(x)\right\} \mid^2}{\int\left[\mathcal{D}g_{ij}^{\perp}(x)\right] \mid \Psi\left\{g_{ij}^{\perp}\right\} \mid^2}, \tag{35}$$

$$\bar{g}_{ij}^{\perp}(x)\ \bar{g}_{kl}^{\perp}(x) + K_{ijkl}^{\perp}\left(\overrightarrow{x}, \overrightarrow{y}\right) = \frac{\int\left[\mathcal{D}g_{ij}^{\perp}(x)\right]\int d^3x g_{ij}^{\perp}(x)\, g_{kl}^{\perp}(y) \mid \Psi\left\{g_{ij}^{\perp}(x)\right\} \mid^2}{\int\left[\mathcal{D}g_{ij}^{\perp}(x)\right] \mid \Psi\left\{g_{ij}^{\perp}\right\} \mid^2}. \tag{36}$$

It follows that

$$\int\left[\mathcal{D}h_{ij}^{\perp}(x)\right]\left(g_{ij}^{\perp}(x) - \bar{g}_{ij}^{\perp}(x)\right) \mid \Psi\left\{g_{ij}^{\perp}(x)\right\} \mid^2 = 0$$

by traslation invariance of the measure

$$\int \left[\mathcal{D}h_{ij}^{\perp}(x)\right] h_{ij}^{\perp}(x) \mid \Psi\left\{g_{ij}^{\perp}(x)+\bar{g}_{ij}^{\perp}(x)\right\} \mid^{2}=0$$

$$\Longrightarrow \int \left[\mathcal{D}h_{ij}^{\perp}(x)\right] h_{ij}^{\perp}(x) \mid \Psi\left\{h_{ij}^{\perp}(x)\right\} \mid^{2}=0 \tag{37}$$

and (36) becomes

$$\begin{aligned} &\int \left[\mathcal{D}h_{ij}^{\perp}(x)\right] \int d^{3}x h_{ij}^{\perp}(x) h_{kl}^{\perp}(y) \mid \Psi\left\{h_{ij}^{\perp}(x)\right\} \mid^{2}= \\ &K_{ijkl}^{\perp}\left(\overrightarrow{x},\overrightarrow{y}\right) \int \left[\mathcal{D}h_{ij}^{\perp}(x)\right] \mid \Psi\left\{h_{ij}^{\perp}\right\} \mid^{2} \end{aligned} \tag{38}$$

Rather than applying the variational principle arbitrarily, the Gaussian *Ansatz* is made, where in the beginning of this calculus one has to replace previous general formulas with

$$\Psi_{\alpha}\left[h_{ij}\left(\overrightarrow{x}\right)\right]=\mathcal{N}\exp\left\{-\frac{1}{4l_{p}^{2}}\left\langle(g-\bar{g})K^{-1}(g-\bar{g})\right\rangle_{x,y}^{\perp}+\ldots\ldots\right\}. \tag{39}$$

With this choice and with formulas (37, 38), the one loop-like Hamiltonian can be written as

$$H^{\perp}=\frac{1}{4l_{p}^{2}}\int_{\mathcal{M}} d^{3}x\sqrt{g}G_{\alpha}^{ijkl}\left[K^{-1\perp}(x,x)_{ijkl}+(\triangle_{2})_{j}^{a}K^{\perp}(x,x)_{iakl}\right] \tag{40}$$

where the first term in square brackets comes from the kinetic part and the second comes from the expansion of the ${}^{3}R$ up to second order in such a way to obtain a quantum harmonic oscillator equation type. the Green function $K^{\perp}(x,x)_{iakl}$ can be represented as

$$K^{\perp}(x,x)_{iakl}:=\sum_{N}\frac{h_{ia}(x)h_{kl}(y)}{2\lambda_{N}(p)}, \tag{41}$$

where $h_{ia}(x)$ are the eigenfunctions of $\triangle_{2j}^{a}$ and $\lambda_{N}(p)$ are infinite variational parameters. In formula (40) we have written the Spin-2 contribution to the energy density alone; expressions like (40) exist for Spin-1 and Spin-0 terms of $\mathcal{H}$.

V. THE SPECTRUM OF THE SPIN-2 OPERATOR AND THE EVALUATION OF THE ENERGY DENSITY

The Spin-2 operator is defined by:

$$\triangle_2 := -\triangle + 2Ric \tag{42}$$

or in components,

$$(\triangle_2)^a_j := -\triangle\delta^a_j + 2R^a_j \tag{43}$$

where $\triangle$ is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and

R^a_j is the mixed Ricci tensor whose components are:

$$R^a_j = diag\left\{\frac{-2m}{r^3}, \frac{m}{r^3}, \frac{m}{r^3}\right\}, \tag{44}$$

where $2m = 2MG$ This operator is similar to the Lichnerowicz operator provided that we substitute the Riemann tensor with the Ricci tensor. In (42) or (43) Ricci tensor acts as a potential on the space of TT tensors; for this reason we are led to study the following eigenvalue equation

$$\left(-\triangle\delta^a_j + 2R^a_j\right) h^i_a = E^2 h^i_j \tag{45}$$

where E^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation into modes of definite frequency, angular momentum and parity. In this paper we are interested to positive E^2 and low lying states with $L = M = 0$, where L is the quantum number corresponding to the square of angular momentum and M is the quantum number corresponding to the projection of the angular momentum on the z-axis. For $L = 0$, Regge-Wheeler decomposition [7] shows that there are no odd-parity perturbations at all, therefore:

$$h^{even}_{ij} = diag\left[H(r)\left(1 - \frac{2m}{r}\right)^{-1}, r^2 K(r), r^2 \sin^2\vartheta K(r)\right] Y_{00}(\vartheta, \phi)\, e^{-i\varpi t} \tag{46}$$

with t fixed. The representation (46) is very useful, because of the decoupling of the modes. This particular situation is valid only for S-wave,

$$-\triangle H(r) - \tfrac{4m}{r^3} H(r) = E^2 H(r)$$

$$-\triangle K(r) + \tfrac{2m}{r^3} K(r) = E^2 K(r) \tag{47}$$

$$-\triangle K(r) + \tfrac{2m}{r^3} K(r) = E^2 K(r)$$

The Laplacian in this particular geometry can be written as

$$\triangle = \left(1 - \frac{2m}{r}\right)\frac{d^2}{dr^2} + \left(\frac{2r-3m}{r^2}\right)\frac{d}{dr}. \tag{48}$$

Defining reduced fields, such as:

$$H(r) = \frac{h(r)}{r}; K(r) = \frac{k(r)}{r}, \tag{49}$$

and changing variables to

$$x = 2m\left\{\sqrt{\frac{r}{2m}}\sqrt{\frac{r}{2m}-1} + \ln\left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m}-1}\right)\right\}, \tag{50}$$

the system (47) becomes

$$-\tfrac{d^2}{dx^2} h(x) - V(x) h(x) = E^2 h(x)$$

$$-\tfrac{d^2}{dx^2} k(x) + V(x) k(x) = E^2 k(x) \tag{51}$$

$$-\tfrac{d^2}{dx^2} k(x) + V(x) k(x) = E^2 k(x)$$

where

$$V(x) = \frac{3m}{r^3} \tag{52}$$

We note that the new variable is such that

$$x \simeq r \; r \longrightarrow \infty \; V(x) \longrightarrow 0$$

$$\tag{53}$$

$$x \simeq 0 \; r \longrightarrow r_0 \; V(x) \longrightarrow \tfrac{3m}{(r_0)^3} = const,$$

where r_0 is the wormhole radius, satisfying the condition $r_0 > 2m$, strictly. The solution of (51), in both cases (flat and curved one) is a Bessel function and precisely the spherical Bessel function of the first kind for the $L = 0$ value of the angular momentum

$$j_0\left(pr\right) = p\sqrt{\frac{2}{\pi}}\frac{\sin\left(pr\right)}{pr} = \sqrt{\frac{2}{\pi}}\frac{\sin\left(pr\right)}{r} \tag{54}$$

The corresponding Green function for this problem will be

$$K\left(x,y\right) = \frac{j_0\left(px\right)j_0\left(py\right)}{2\lambda}\cdot\frac{1}{4\pi} \tag{55}$$

Substituting (55)in(40) one gets (after normalization in spin space and after a rescaling of the fields in such a way to absorb l_p^2)

$$E\left(m,\lambda\right) = \frac{V}{2\pi^2}\sum_{i=1}^{2}\int_0^\infty dpp^2\left[\lambda_i\left(p\right) + \frac{E_i^2\left(p,m\right)}{\lambda_i\left(p\right)}\right] \tag{56}$$

where

$$E_{1,2}^2\left(p,m\right) = p^2 \mp \frac{3m}{r_0^3}, \tag{57}$$

$\lambda_i\left(p\right)$ are variational parameters corresponding to the eigenvalues for a (graviton) spin-2 particle in an external field and V is the volume of the system.

By minimizing (56) with respect to $\lambda_i\left(p\right)$ one obtains $\overline{\lambda}_i\left(p\right) = \left[E_i^2\left(p,m\right)\right]^{\frac{1}{2}}$ and

$$E\left(m,\overline{\lambda}\right) = \frac{V}{2\pi^2}\sum_{i=1}^{2}\int_0^\infty dp2\sqrt{E_i^2\left(p,m\right)} \text{ with } p^2 > \frac{3m}{r_0^3} \tag{58}$$

The total energy in the presence of the background is

$$E\left(m\right) = \frac{V}{2\pi^2}\frac{1}{2}\int_0^\infty dpp^2\left(\sqrt{p^2-c^2} + \sqrt{p^2+c^2}\right) \text{ where } c^2 = \frac{3m}{r_0^3} \tag{59}$$

For flat space the calculation is essentially the same with the exception of $c^2 = 0$. Therefore the equivalent of (59) in flat space is

$$E\left(0\right) = \frac{V}{2\pi^2}\frac{1}{2}\int_0^\infty dpp^2\left(2\sqrt{p^2}\right) \tag{60}$$

Now, we are in position to perform the energy difference between (59)and (60). $\Delta E\left(m\right)$ up to second order in perturbations is

$$\Delta E\left(m\right)=\frac{V}{2\pi^{2}}\frac{1}{2}\int_{0}^{\infty}dpp^{2}\left[\sqrt{p^{2}-c^{2}}+\sqrt{p^{2}+c^{2}}-2\sqrt{p^{2}}\right] \tag{61}$$

We want to evaluate the UV behaviour of (61), therefore

$$\Delta E\left(m\right)=\tfrac{V}{2\pi^{2}}\tfrac{1}{2}\int_{0}^{\infty}dpp^{3}\left[\sqrt{1-\left(\tfrac{c}{p}\right)^{2}}+\sqrt{1+\left(\tfrac{c}{p}\right)^{2}}-2\right]$$

becomes for $p^{2}>>c^{2}$

$$\sim\tfrac{V}{2\pi^{2}}\tfrac{1}{2}\int_{0}^{\infty}dpp^{3}\left[1-\tfrac{1}{2}\left(\tfrac{c}{p}\right)^{2}-\tfrac{1}{8}\left(\tfrac{c}{p}\right)^{4}+1+\tfrac{1}{2}\left(\tfrac{c}{p}\right)^{2}-\tfrac{1}{8}\left(\tfrac{c}{p}\right)^{4}-2\right]$$

$$=-\frac{V}{2\pi^{2}}\frac{c^{4}}{8}\int_{0}^{\infty}\frac{dp}{p} \tag{62}$$

Introducing a cut-off one gets for the UV limit

$$\int_{0}^{\infty}\frac{dp}{p}\sim\int_{0}^{\frac{\Lambda}{c}}\frac{dx}{x}\sim\ln\left(\frac{\Lambda}{c}\right) \tag{63}$$

and $\Delta E\left(m\right)$ for high momenta can be estimated by the following expression

$$\Delta E\left(m\right)\sim-\frac{V}{2\pi^{2}}\frac{c^{4}}{16}\ln\left(\frac{\Lambda^{2}}{c^{2}}\right)=-\frac{V}{2\pi^{2}}\left(\frac{3m}{r_{0}^{3}}\right)^{2}\frac{1}{16}\ln\left(\frac{r_{0}^{3}\Lambda^{2}}{3m}\right). \tag{64}$$

At this point we can compute the total energy, namely the classical contribution plus the quantum correction up to second order. Recalling the definition of asymptotic energy for an asymptotically flat background, like the Schwarzschild one

$$E_{\mathcal{ADM}}=\lim_{r\rightarrow\infty}\int_{\partial\mathcal{M}}\sqrt{\hat{g}}\hat{g}^{ij}\left[\hat{g}_{ik,j}-\hat{g}_{ij,k}\right]dS^{k}, \tag{65}$$

where $\hat{g}_{ij}$ is the metric induced on a spacelike hypersurface $\partial\mathcal{M}$ which has a boundary at infinity like S^{2}, one gets,

$$M-\frac{V}{2\pi^{2}}\left(\frac{3m}{r_{0}^{3}}\right)^{2}\frac{1}{16}\ln\left(\frac{r_{0}^{3}\Lambda^{2}}{3m}\right)=M-\frac{V}{2\pi^{2}}\left(\frac{3MG}{r_{0}^{3}}\right)^{2}\frac{1}{16}\ln\left(\frac{r_{0}^{3}\Lambda^{2}}{3MG}\right) \tag{66}$$

One can observe that

$$\Delta E\left(m\right)\rightarrow\infty\text{ when }m\rightarrow0,\text{ for }r_{0}=2m=2GM \tag{67}$$

and

$$\Delta E\left(m\right)\rightarrow0\text{ when }m\rightarrow0,\text{ for }r_{0}\neq2m=2GM \tag{68}$$

VI. SUMMARY AND CONCLUSIONS

The trial wave functional approach, by means of Gaussian configurations, led to possible calculations about quantum fluctuations of the gravitational field around some fixed background geometry, i.e. Schwarzschild. In particular, with the choice (10) one is able, momentarily, to avoid the constraint equations typical of the hamiltonian formulation of general relativity and to bypass the problem of "*time*". With this gauge choice the problem of defining a correct vacuum energy is well posed. As we can see, the result shows us an intrinsic energy depending only on the dynamics generated by 3-surfaces. A little comment is necessary to explain the reasons that support the results of formula (64). In that formula we introduced a particular value of the radius, which behaves as a regulator with respect to the horizon approach of the potential. The meaning of this particular value is related to the necessity of explaining the dynamical origin of black hole entropy by the entanglement entropy mechanism and by the so-called "*brick wall model*" [6]. Indeed, the same mechanism is present when one has to regularize entropy by imposing a kind of cut-off, that in coordinate space means $r_0 > 2m$. It is known that at one-loop level Gravity is renormalizable only in flat space. In a dimensional regularization scheme its contribution to the action is, on shell, proportional to the Euler character of the manifold that is nonzero for the instanton. To deal with this one must introduce a regulator that indeed appears in the contribution of the energy density.

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APPENDIX A: CONVENTIONS

Here we give the conventions for the metric tensor, connections and the curvature tensor:

1. Background Metric

$$g_{11} = \frac{1}{1-\frac{2m}{r}},\ g_{22} = r^2,\ g_{33} = r^2 \sin^2\theta$$

$$g^{11} = 1 - \frac{2m}{r},\ g^{22} = r^{-2},\ g^{33} = r^{-2}\sin^{-2}\theta \tag{A1}$$

2. Connection

$$\Gamma^1_{ab} = \begin{pmatrix} -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1} & & \\ & -\left(1-\frac{2m}{r}\right) r & \\ & & -\left(1-\frac{2m}{r}\right) r\sin^2\theta \end{pmatrix}$$

$$\Gamma^2_{ab} = \begin{pmatrix} 0 & r^{-1} & 0 \\ r^{-1} & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}\ \Gamma^3_{ab} = \begin{pmatrix} 0 & 0 & r^{-1} \\ 0 & 0 & \cot\theta \\ r^{-1} & \cot\theta & 0 \end{pmatrix} \tag{A2}$$

3. Riemann tensor, Ricci tensor and the Scalar Curvature in $3D$

$$R^l_{ijm} = \Gamma^l_{mi,j} - \Gamma^l_{ji,m} + \Gamma^l_{ja}\Gamma^a_{mi} - \Gamma^l_{ma}\Gamma^a_{ji} \quad \text{Riemann tensor}$$

Because of the vanishing of the Weyl tensor in $3D$, that is $C^l_{ijm} = 0$, Riemann tensor is completely determined by Ricci tensor

$$R_{lijm} = g_{lj}R_{im} - g_{lm}R_{ij} - g_{ij}R_{lm} + g_{im}R_{lj}$$

$$R_{im} = R^l_{ilm} \quad \text{Ricci tensor}$$

$$R = g^{lj}R_{lj} \quad \text{Scalar curvature}$$

APPENDIX B: SCALAR CURVATURE EXPANSION

In this part we give the necessary tools for the scalar curvature expansion in terms of the fluctuations of 3-surfaces around the background geometry. Metric tensor will be separated in a classical part (background) plus a quantum part i.e.

$$g_{ij} = \bar{g}_{ij} + h_{ij} \tag{B1}$$

1. Expansion of the determinant

$$\sqrt{g_{ij}} = \exp Tr \ln \sqrt{g_{ij}} = \exp Tr \tfrac{1}{2} \ln \left(\bar{g}_{ij} + h_{ij}\right) = \exp Tr \tfrac{1}{2} \left[\ln \bar{g}_{ij} \left(1 + \tfrac{h_{ij}}{\bar{g}_{ij}}\right)\right] =$$

$$\exp Tr \tfrac{1}{2} \left[\ln \bar{g}_{ij} + \ln \left(1 + \tfrac{h_{ij}}{\bar{g}_{ij}}\right)\right] \simeq \tag{B2}$$

$$\sqrt{\bar{g}_{ij}} \left(1 + \tfrac{1}{2} h - \tfrac{1}{4} h^i_k h^k_i + \tfrac{1}{6} h^i_k h^k_l h^l_i - \tfrac{1}{8} h h^i_k h^k_i + \tfrac{1}{8} h^2 + \tfrac{1}{8} h^3\right) + O\left(h^4\right)$$

2. The inverse metric expansion

$$g^{ij} = \frac{\bar{g}^i_k}{\delta^j_k + h^j_k} \simeq \bar{g}^{ij} - h^{ij} + h^{ik} h^j_k - h^{ik} h^l_k h^j_l + h^{ik} h^l_k h^m_l h^j_m + O\left(h^5\right) \tag{B3}$$

3. Connection

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left(g_{li,j} + g_{lj,i} - g_{ij,l}\right)$$

To 0^{th} order $\Gamma^k_{ij} = \Gamma^{k(0)}_{ij} = \frac{1}{2} \bar{g}^{kl} \left(\bar{g}_{li,j} + \bar{g}_{lj,i} - \bar{g}_{ij,l}\right)$.

To 1^{st} order $\Gamma^k_{ij} = S^k_{ij} = \frac{1}{2} \bar{g}^{kl} \left(h_{li|j} + h_{lj|i} - h_{ij|l}\right)$

The higher order corrections to the connection are related by the formula

$$\Gamma^{k(n)}_{ij} = -h^k_l \Gamma^{l(n-1)}_{ij} \text{ where } \Gamma^{k(1)}_{ij} = S^k_{ij} \tag{B4}$$

4. Riemann Tensor

$$R^l_{ijm} = \Gamma^l_{mi,j} - \Gamma^l_{ji,m} + \Gamma^l_{ja} \Gamma^a_{mi} - \Gamma^l_{ma} \Gamma^a_{ji}$$

It is convenient to divide Riemann tensor into two terms: linear and non-linear

$$Lin\left(R^l_{ijm}\right) = L^l_{ijm} := \Gamma^l_{mi,j} - \Gamma^l_{ji,m}$$

$$N - Lin\left(R^l_{ijm}\right) = N^l_{ijm} := \Gamma^l_{ja}\Gamma^a_{mi} - \Gamma^l_{ma}\Gamma^a_{ji} \tag{B5}$$

The higher order terms of L^l_{ijm} are simply

$$L^{l(n)}_{ijm} := \Gamma^{l(n)}_{mi|j} - \Gamma^{l(n)}_{ji|m} \tag{B6}$$

while higher orders of N^l_{ijm} are

$$N^{l(n)}_{ijm} := \sum_{j=1}^{n-1}\left[\Gamma^{a(j)}_{mi}\Gamma^{l(n-j)}_{ja} - \Gamma^{l(n-j)}_{ma}\Gamma^{a(j)}_{ji}\right] \tag{B7}$$

5. Second order scalar curvature

Collecting together previous expansion formulas we can write the following expression for the ${}^3R$ scalar curvature expanded up to second order:

$$\int d^3x\left[-\frac{1}{4}h\triangle h + \frac{1}{4}h^{li}\triangle h_{li} - \frac{1}{2}h^{ij}\nabla_l\nabla_i h^l_j + \frac{1}{2}h\nabla_l\nabla_i h^{li} - \frac{1}{2}h^{ij}R_{aj}h^a_j + \frac{1}{2}hR_{ij}h^{ij}\right]. \tag{B8}$$

REFERENCES

[1] L. J. Garay, Phys. Rev. D **48,** 1710 (1993).

[2] P. O. Mazur and E. Mottola, Nucl. Phys. **B 341**, 187 (1990), D. Giulini, Phys. Rev. D **10** 5630 (1995).

[3] J. W. York Jr., J. Math. Phys., **14**, 4 (1973).

[4] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (John Wiley & Sons, Inc., New York, 1962); B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[5] A. K. Kerman and D. Vautherin, Ann. Phys., **192**, 408 (1989); J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D **8**, 2428 (1974); R. Jackiw in *Séminaire de Mathématiques Supérieures, Montréal, Québec, Canada- June 1988 - Notes by P. de Sousa Gerbert*; M. Consoli and G. Preparata, Phys. Lett. B, **154**, 411 (1985).

[6] G.'t Hooft, Nucl. Phys. B **256** (1985), 727; V.P. Frolov, I.Novikov, Phys. Rev. D **48** (1993), 4545.

[7] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957)